Basic symplectic geometry for p-branes with thickness in a curved background

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Abstract

We show that the Witten covariant phase space for p-branes with thickness in an arbitrary background is endowed of a symplectic potential, which although is not important to the dynamics of the system, plays a relevant role on the phase space, allowing us to generate a symplectic structure for the theory and therefore give a covariant description of canonical formalism for quantization.

I. INTRODUCTION

As we know, a covariant description of the canonical formalism for quantization and the study of the symmetry aspects has been given by using basic ideas of symplectic geometry. This formalism, has all virtues that Feyman's path integral has, that is, manifestly covariant, maintaining all relevant symmetries, such as Poincaré invariance. This scheme, also has been applied in many theories, for example, Witten *et al* take the cases of Yang-Mills and General Relativity [1], open superstrings [2], and the analysis of diffeomorphism invariant field theories was considered by Wald *et al* [3], among others. Recently this formalism was taken up by Cartas-Fuentevilla to p-branes governed by the Dirac-Nambu-Goto action [DNG] [4], using a weakly covariant formalism for deformations, introduced by Capovilla-Guven [CG] [5], and in [6] using a strongly covariant formalism, introduced by Carter [7].

On the other hand, in many cases it was seen that [DNG] action is inadequate and there are missing corrective quadratic terms in the extrinsic curvature. For example, in the eighties Polyakov proposed a modification

to the [DNG] action by adding a rigidity term constructed with the extrinsic curvature of the worldsheet generated by a string, and to include quadratic terms in the extrinsic curvature to the [DNG] action is absolutely necessary, because of its influence in the infrared region determines the phase structure of the string theory, in this manner, we can compute the critical behavior of random surfaces an their geometrical and physical characteristics [8]. In the treatment of topological defects [9], curvature terms are induced by considering an expansion in the thickness of the defect. Bosseau and Letelier have studied cosmic strings with arbitrary curvature corrections, finding for example, that the curvature correction may change the relation between the string energy density and the tension [10]. Furthermore, such models have been used to describe mechanical properties of lipid membranes [11]. More recently, conservation laws for bosonic brane dynamics have been obtained for an action quadratic in the extrinsic curvature [12].

Due to the above ideas, the purpose of this article is to establish the bases of the covariant canonical formalism for corrections to the [DNG] action, which depend quadratically on extrinsic curvature (we will denote this corrective term by [QEC]).

This paper is organized as follows. In Sect.II, we start with the formalism of deformations introduced by [CG] [7], and we give some remarks for [DNG] p-branes, obtaining by another way the results obtained by Cartas-Fuentevilla [4]. In Sect.III, from tangential deformations, we obtain a symplectic potential for [QEC] action, and the linearized equations of motion taking as special case a extremal surface in an arbitrary background which will be useful in the next section. In Sect.IV, we define the Witten covariant space phase for [QEC] theory, and considering the linearized equations for [QEC] action, we obtain a covariant conserved current by applying the self-adjoin operators method. In Sect.V, we find a two-form for [QEC] theory and we show that is an exact and no-degenerate differential form, from the global potential, found in the Sect. III. In Sect. VI we establish some remarks and prospects.

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II. Global sympletic potential for Dirac-Nambu-Goto action

In [6], Cartas-Fuentevilla showed using a strongly covariant formalism, that the [DNG] action has a covariant conserved symplectic current obtained from a global symplectic potential. In the same way, we shall show that in the weakly covariant formalism used in the present treatment [5] there exists also a global symplectic potential for [DNG] action, from which we will get by another way the symplectic current obtained by Cartas-Fuentevilla in [4].

To prove it, we take the [DNG] action, that is proportional to the area of the spacetime trajectory created

by the brane,

$$S = -\mu \int \sqrt{-\gamma} d^D \xi, \tag{1}$$

where μ is the brane tension. In agreement with [5], we take the tangential and normal deformations of the action (1), and we obtain

$$\delta S = -\mu \int \sqrt{-\gamma} \nabla_a \Phi^a d^D \xi - \mu \int K^i \phi_i d^D \xi, \tag{2}$$

where

$$\phi^a = \delta X^\mu e^a{}_\mu \quad \text{and} \quad \phi^i = \delta X^\mu n^i{}_\mu, \tag{3}$$

δX^μ being the infinitesimal spacetime variation of the embbeding, with n^i and e^a as the vector fields normal and tangent to the worldsheet respectively.

We can see that the second term on the right hand-side of equation (2) corresponds to the equation of motion, $K^i = 0$, and the corresponding linearized equations are [4]

$$[\widetilde{\Delta}^i_j + K_{ab}{}^i K^{ab}{}_j - g(R(e_a, n_j)e^a, n^i)]\phi^j = 0, \tag{4}$$

where $\widetilde{\Delta} = \widetilde{\nabla}^a \widetilde{\nabla}_a$, $K_{ab}{}^i$ is the extrinsic curvature, and $g(R(e_a, n_j)e^a, n^i) = R_{\alpha\beta\mu\nu} n_j{}^\alpha e_a{}^\beta e^{a\mu} n^{i\nu}$, being $R_{\alpha\beta\mu\nu}$ the background Riemann tensor (for more detail see the Appendix).

On the other hand, the argument of the total divergence, ϕ^a, given in (2), are neglected in the literature because of is not relevant locally to the dynamics of the system. However, we will identify $-\sqrt{-\gamma}\phi^a$ from the first term on the right-hand side in equation (2) as a symplectic potential on the phase space and we will take its variation (its exterior derivative on Z, see the Appendix, equation (55)), this is

$$D_\delta(-\sqrt{-\gamma}\phi^a) = \sqrt{-\gamma}[K^{abi}\phi_i\phi_b + \phi_i \widetilde{\nabla}^a \phi^i]. \tag{5}$$

It is important to notice that because of ϕ^a is a diffeomorphism on the world-sheet, it can be gauged away in the equation (5), although its variation does not vanish, thus

$$D_\delta(-\sqrt{-\gamma}\phi^a) = \sqrt{-\gamma}[\phi_i \widetilde{\nabla}^a \phi^i], \tag{6}$$

in this manner, we can see that the last equation is the symplectic current obtained by Cartas-Fuentevilla [4] applying the self-adjoint operators method. Thus, equation (6) implies that the symplectic structure obtain in [4] is no only a closed two-form but even an exact two-form.

Therefore, we can identify indeed $-\sqrt{-\gamma}\phi^a$ as a global symplectic potential for [DNG] p-branes, that can not be neglected because of allow us construct geometrical structures physically relevant on the phase space Z. Following these ideas, in the next section we shall consider an action quadratic in the extrinsic curvature.

III.The quadratic term in the extrinsic curvature

Let us consider the following action quadratic in the extrinsic curvature

$$S_2 = \alpha \int d^D\xi \sqrt{-\gamma} K_i K^i, \tag{7}$$

where

$$K^i = \gamma^{ab} K_{ab}{}^i, \tag{8}$$

and α is the brane tension coefficient. As in the last section, and using the [CG] deformation formalism [5], we take the tangential and normal deformations of the world-volume and we obtain,

$$\begin{aligned} \delta S_2 \;&=\; 2\alpha \int d^D\xi \sqrt{-\gamma}\left[-\widetilde{\triangle} K^i \phi_i + \left(g(R(e_a, n^j)e^a, n^i) - (\gamma^{ac}\gamma^{bd} - \frac{1}{2}\gamma^{ab}\gamma^{cd}) K_{ab}{}^j K_{cd}{}^i\right)\phi_i K_j\right] \\ &+\; 2\alpha \int d^D\xi \nabla_a \left[\sqrt{-\gamma}\left(\frac{1}{2}K^j K_j \Phi^a + \phi_i \widetilde{\nabla}^a K^i - K_i \widetilde{\nabla}^a \phi^i\right)\right], \end{aligned} \tag{9}$$

where we can find the equations of motion

$$\widetilde{\triangle} K^i + \left(-g(R(e_a, n^j)e^a, n^i) + (\gamma^{ac}\gamma^{bd} - \frac{1}{2}\gamma^{ab}\gamma^{cd}) K_{ab}{}^j K_{cd}{}^i\right) K_j = 0, \tag{10}$$

and we identify from the pure divergence term in (9), the following

$$\Psi^a = \sqrt{-\gamma}\left[\frac{1}{2}K^j K_j \phi^a + \phi_i \widetilde{\nabla}^a K^i - K_i \widetilde{\nabla}^a \phi^i\right], \tag{11}$$

as a symplectic potential for [QEC] p-branes, that are neglected, because of does not contribute locally to the dynamics of system, but it generates a geometrical structure on the phase space, as we will see in the next section.

We can obtain the linearized equations taking the variation of the equation (10), which are:

$$\begin{aligned} &-\; \widetilde{\Delta}\widetilde{\Delta}\phi^i - 2K^{ab}{}_j K^j (\widetilde{\nabla}_a \widetilde{\nabla}_b \phi^i) + \frac{1}{2}K^j K_j \widetilde{\Delta}\phi^i + (K^i K_j - 2K_{ab}{}^i K^{ab}{}_j)\widetilde{\Delta}\phi^j \\ &-\; 2K^{ab}{}_j (\widetilde{\nabla}_a K^j)(\widetilde{\nabla}_b \phi^i) - K_j (\widetilde{\nabla}_a K^{abj})(\widetilde{\nabla}_b \phi^i) - 2K^{ab}{}_j (\widetilde{\nabla}_a K^i)(\widetilde{\nabla}_b \phi^j) \\ &-\; 2\widetilde{\nabla}^c [K_{ab}{}^i K^{ab}{}_j](\widetilde{\nabla}_c \phi^j) + 2K^{abi}(\widetilde{\nabla}_a K_j)(\widetilde{\nabla}_b \phi^j) - \widetilde{\Delta}[K_{ab}{}^i K^{ab}{}_j]\phi^j \\ &+\; K_j (\widetilde{\nabla}^b K^i)(\widetilde{\nabla}_b \phi^j) + K_j (\widetilde{\nabla}_a K^{abi})(\widetilde{\nabla}_b \phi^j) + \widetilde{\nabla}_a K_j \widetilde{\nabla}^a K^i \phi^j - 2(\widetilde{\nabla}_b K^i)(\widetilde{\nabla}_a K^{ab}{}_j)\phi^j \\ &-\; 2K^{ab}{}_j (\widetilde{\nabla}_a \widetilde{\nabla}_b K^i)\phi^j + 2K_{ab}{}^i K^{bc}{}_k K^a{}_{cj} K^j \phi^k + \frac{1}{2}K_{ab}{}^i K^{ab}{}_k K^j K_j \phi^k + K^i K_j K_{ab}{}^j K^{ab}{}_k \phi^k \\ &-\; K_{ab}{}^i K^{abj} K_{cdj} K^{cd}{}_k \phi^k - g(R(n_j, e^b,)n^k, n^i)\phi^j \widetilde{\nabla}_b K_k - \widetilde{\nabla}_b [g(R(n_j, e^b,)n^k, n^i))\phi^j K_k] \\ &+\; K^{cdi} g(R(e_c, n_l)e_d, n^j)\phi^l K_j + K^{cdj} g(R(e_c, n_l)e_d, n^i)\phi^l K_j + K^{cdj} K_{cd}{}^i g(R(e_a, n_l)e^a, n_j)\phi^l \end{aligned}$$

$$
\begin{aligned}
&- \; K_j K^i g(R(e_a,n_l)e^a,n^j)\phi^l - \frac{1}{2} g(R(e_a,n_j)e^a,n^i)\phi^j K_l K^l + g(R(e_a,n^l)e^a,n^i)K_{cdl}K^{cdj}\phi_j \\
&+ \; \widetilde{\Delta}[g(R(e_a,n^j)e^a,n^i)]\phi_j + 2g(R(e_a,n^j)e^a,n^i)\widetilde{\Delta}\phi_j + 2\widetilde{\nabla}_a[g(R(e_b,n^j)e^b,n^i)]\widetilde{\nabla}^a\phi_j \\
&- \; g(R(e_a,n^l)e^a,n^i)g(R(e_b,n_j)e^b,n_l)\phi^j - K_j\delta[g(R(e_b,n^j)e^b,n^i)] = 0,
\end{aligned} \tag{12}
$$

where equations (4.6) and (4.16) of [5] have been employed.

For simplicity we set $K^i = 0$ (extremal surfaces) in the linearized equations (12), then the equation is reduced to

$$
-(P^2)^i{}_j\phi^j = 0, \tag{13}
$$

where the operator P^i_j is given by,

$$
P^i{}_j = \left[\widetilde{\triangle}^i_j + K_{ab}{}^i K^{ab}{}_j - g(R(e_b,n_j)e^b,n^i)\right]. \tag{14}
$$

We can see that the equation (14), is equal to the operator of the linearized equations for [DNG] action, equation (4), which describes the deformations of extremal surfaces in a curved background.

Writing (13) explicitly, we find that

$$
\begin{aligned}
(P^2)^i{}_j\phi^j \; &= \; \widetilde{\triangle}\widetilde{\triangle}\delta^i_j\phi^j + 2K_{ab}{}^iK^{ab}{}_j\widetilde{\triangle}\phi^j + 2\tilde{\nabla}^c\left[K_{ab}{}^iK^{ab}{}_j\right](\widetilde{\nabla}_c\phi^j) + \widetilde{\triangle}\left[K_{ab}{}^iK^{ab}{}_j\right]\phi^j \\
&+ \; K_{ab}{}^iK^{abk}K_{cdk}K^{cd}{}_j\phi^j - 2g(R(e_b,n^j)e^b,n^i)\widetilde{\Delta}\phi_j - \widetilde{\Delta}[g(R(e_b,n^j)e^b,n^i)]\phi_j \\
&- \; K_{abj}K^{abl}g(R(e_b,n^j)e^b,n^i)\phi_l - K^{cd}{}_jK_{cd}{}^ig(R(e_b,n_l)e^b,n^j)\phi^l \\
&- \; 2\widetilde{\nabla}^c[g(R(e_b,n^j)e^b,n^i)]\widetilde{\nabla}_c\phi_j + g(R(e_b,n^j)e^b,n^i)g(R(e_b,n_l)e^b,n_j)\phi^l = 0.
\end{aligned} \tag{15}
$$

It is remarkable that the solutions of the perturbations about an extremal surface for [DNG] action, equation (4), continue to being solutions for [QEC], equation (15), even existing a curved background. This is a more general result than that presented in [7], for a flat spacetime.

IV. Self-adjointness of the operators governing the deformations

In this section, we shall show that the operator $(P^2)^i{}_j$ in the equation (14) is self-adjoint, which guarantees that a symplectic current can be constructed in terms of solutions of equation (15). For beginning, following [13] we define $M^i{}_j = M_j{}^i \equiv -K_{ab}{}^iK^{ab}{}_j + g(R(e_b,n_j)e^b,n^i)$ as the mass matrix, therefore we can rewrite the equation (15) as follows

$$
\begin{aligned}
[\widetilde{\triangle}\widetilde{\triangle}\delta^i_j \; &- \; 2M^i{}_j\widetilde{\triangle} - 2\tilde{\nabla}^c[M^i{}_j](\widetilde{\nabla}_c) - \widetilde{\triangle}[M^i{}_j] \\
&+ \; M^{ik}M_{kj}]\phi^j = 0.
\end{aligned} \tag{16}
$$

Now, let ϕ_1^i and ϕ_2^i be two arbitrary scalar fields, which correspond to a pair of solutions of equation (15), thus is very easy to prove the following,

$$\phi_{1i}\widetilde{\triangle}\widetilde{\triangle}\phi_2^i \equiv (\widetilde{\triangle}\widetilde{\triangle}\phi_{1i})\phi_2^i + \nabla_a {j_1}^a, \tag{17}$$

where ${j_1}^a$ is given by

$${j_1}^a = \phi_{1i}\widetilde{\nabla}^a\widetilde{\triangle}\phi_2^i + \widetilde{\triangle}\phi_{1i}\widetilde{\nabla}^a\phi_2^i - \widetilde{\nabla}^a\phi_{1i}\widetilde{\triangle}\phi_2^i - \widetilde{\nabla}^a\widetilde{\triangle}\phi_{1i}\phi_2^i. \tag{18}$$

Furthermore, we can demonstrate that,

$$-2{M^i}_j(\phi_{1i}\widetilde{\triangle}\phi_2^j) - 2\widetilde{\nabla}_a{M^i}_j(\phi_{1i}\widetilde{\nabla}^a\phi_2^j) \equiv -2{M^i}_j(\widetilde{\triangle}\phi_{1i})\phi_2^j - 2\widetilde{\nabla}_a{M^i}_j(\widetilde{\nabla}^a\phi_{1i})\phi_2^j + \nabla_a j_2^a, \tag{19}$$

with ${j_2}^a$:

$${j_2}^a = -2\left[{M^i}_j\phi_{1i}\widetilde{\nabla}^a\phi_2^j - {M^i}_j\widetilde{\nabla}^a\phi_{1i}\phi_2^j\right], \tag{20}$$

finally we obtain, putting (18) and (20) together

$$\begin{aligned}
\phi_{1i} \quad & [\widetilde{\triangle}\widetilde{\triangle}\delta^{ij} - 2M^{ij}\widetilde{\triangle} - 2\tilde{\nabla}^c[M^{ij}](\widetilde{\nabla}_c) - \widetilde{\triangle}[M^{ij}] + M^{ik}M^{kj}]\phi_{2j} \\
= \quad & [[\widetilde{\triangle}\widetilde{\triangle}\delta^{ji} - 2M^{ji}\widetilde{\triangle} - 2\tilde{\nabla}^c[M^{ji}](\widetilde{\nabla}_c) - \widetilde{\triangle}[M^{ji}] + M^{jk}M^{ki}]\phi_{1i}]\phi_{2j} \\
+ \quad & \nabla_a j^a,
\end{aligned} \tag{21}$$

where we have considered the symmetry of the mass matrix, and j^a is given by

$$\begin{aligned}
j^a \;=\;& \phi_{1i}\widetilde{\nabla}^a\widetilde{\triangle}\phi_2^i + \widetilde{\triangle}\phi_{1i}\widetilde{\nabla}^a\phi_2^i - \widetilde{\nabla}^a\phi_{1i}\widetilde{\triangle}\phi_2^i - \widetilde{\nabla}^a\widetilde{\triangle}\phi_{1i}\phi_2^i \\
+\;& 2{K_{ab}}^i{K^{ab}}_j\phi_{1i}\widetilde{\nabla}^a\phi_2^j - 2g(R(e_b,n_j)e^b,n^i)\phi_{1i}\widetilde{\nabla}^a\phi_2^j \\
-\;& 2{K_{ab}}^i{K^{ab}}_j\widetilde{\nabla}^a\phi_{1i}\phi_2^j + 2g(R(e_b,n_j)e^b,n^i)\widetilde{\nabla}^a\phi_{1i}\phi_2^j,
\end{aligned} \tag{22}$$

it is remarkable to see that there exist background gravity terms in this expression for our symplectic current. Considering that ϕ_{1i} and ϕ_{2j} correspond to a pair of solutions of the equation (15), equation (21) implies that the operator ${(P^2)^i}_j$ is self-adjoint, therefore, we have

$$\bigtriangledown_a j^a = 0. \tag{23}$$

In the next section, we will take equation (22) on the phase space for [QEC] theory and we will compared with the variation of symplectic potential given in (11).

V. The Witten covariant phase space and the Symplectic Structure on Z

The basic idea of the covariant description of the canonical formalism is to construct a symplectic structure on the classical phase space, instead of choosing $p's$ and $q's$. In this manner, in agreement with [1], the Witten phase space for [DNG-G] theory is the space of solutions of equation (10), that we shall call Z, and on such phase space we will construct a symplectic structure. Thus, we can identify e_a, n^i, $k_{ab}{}^i$, γ_{ab} as zero-forms on Z, and the scalar fields ϕ^i are closed one-forms on Z (see Appendix, section IV), it is

$$\widetilde{D}_\delta \phi^i = 0. \tag{24}$$

In this manner, considering the last paragraph, we can see that the expression (22) is a covariantly conserved two-form on Z. Thus, on the phase space Z it is enough take only one solution [1], then we can set $\phi_{1i} = \phi_{2i} = \phi_i$ in (22), and we obtain without loosing generality

$$\begin{aligned} j^a &= \phi_i \widetilde{\nabla}^a \widetilde{\triangle} \phi^i + \widetilde{\triangle} \phi_i \widetilde{\nabla}^a \phi^i - \widetilde{\nabla}^a \phi_i \widetilde{\triangle} \phi^i - \widetilde{\nabla}^a \widetilde{\triangle} \phi_{1i} \phi^i \\ &+ 2K_{ab}{}^i K^{ab}{}_j \phi_i \widetilde{\nabla}^a \phi^j - 2g(R(e_b, n_j)e^b, n^i)\phi_i \widetilde{\nabla}^a \phi^j \\ &- 2K_{ab}{}^i K^{ab}{}_j \widetilde{\nabla}^a \phi_i \phi^j + 2g(R(e_b, n_j)e^b, n^i)\widetilde{\nabla}^a \phi_i \phi^j, \end{aligned} \tag{25}$$

considering that ϕ_i are one-forms on Z and (hence $\widetilde{\triangle}\phi_i$, $\widetilde{\nabla}^a \widetilde{\triangle}\phi^i$, $\widetilde{\nabla}^a \phi_i$), we have for example $(\widetilde{\nabla}^a \phi^i)\phi_i = -\phi_i(\widetilde{\nabla}^a \phi^i)$, and j^a becomes to be

$$j^a = \phi_i \widetilde{\nabla}^a \widetilde{\Delta} \phi^i + \widetilde{\Delta} \phi^i \widetilde{\nabla}^a \phi_i + 2K_{cd}{}^i K^{cdj} \phi_i \widetilde{\nabla}^a \phi_j - 2g(R(e_b, n^j)e^b, n^i)\phi_i \widetilde{\nabla}^a \phi_j, \tag{26}$$

that we will use in this section. Strictly $\widetilde{\nabla}^a \phi^i \phi_i$ corresponds to the wedge product of one-forms on Z, $\widetilde{\nabla}^a \phi^i \wedge \phi_i$, but in this paper we omit the explicit use of $\wedge$ [see for example [1]].

On the other hand, the symplectic structure on Z is a (non-degenerate) closed two-form; the closeness is equivalent to the Jacobi identity in the conventional Hamiltonian scheme, and the antisymmetry of a two-form represents the antisymmetry of Poisson brackets. In this section, we will find a covariant symplectic structure for [QEC] theory, and we will demonstrate that such a geometric structure is even an exact two-form (which implies that in particular is closed).

To prove the closeness, we shall calculate the variation of Ψ^a. For beginning, we will calculate for an arbitrary field ψ^i the variation $\widetilde{D}_\delta \widetilde{\nabla}_b \psi^i$, this is [5],

$$\begin{aligned} \widetilde{D}_\delta \widetilde{\nabla}_b \psi^i &= D_\delta [D_b \psi^i - \omega_b{}^{ij} \psi_j] - \gamma^{ij} \widetilde{\nabla}_b \psi_j \\ &= D_b D_\delta \psi^i - (D_\delta \omega_b{}^{ij})\psi_j - \omega_b{}^{ij} D_\delta \psi_j - \widetilde{\nabla}_b(\gamma^{ij}\psi_j) + (\widetilde{\nabla}_b \gamma^{ij})\psi_j \\ &= \widetilde{\nabla}_b \widetilde{D}_\delta \psi^i - (D_\delta \omega_b{}^{ij} - \widetilde{\nabla}_b \gamma^{ij})\psi_j. \end{aligned} \tag{27}$$

Using the equations (24) and (27), we have for $\psi^i = \phi^i$,

$$\begin{aligned} \widetilde{D}_\delta \widetilde{\nabla}_b \phi^i &= -(D_\delta {\omega_b}^{ij} - \widetilde{\nabla}_b \gamma^{ij})\phi_j \\ &= {K_{bc}}^i \widetilde{\nabla}^c \phi^j \phi_j - {K_{bc}}^j \widetilde{\nabla}^c \phi^i \phi_j - g(R(n_k, e_b)n^j, n^i)\phi^k \phi_j, \end{aligned} \tag{28}$$

where we have used that [5]

$$D_\delta {\omega_a}^{ij} - \widetilde{\nabla}_a \gamma^{ij} = -{K_{ab}}^i \widetilde{\nabla}^b \phi^j + {K_{ab}}^j \widetilde{\nabla}^b \phi^i + g(R(n_k, e_a)n^j, n^i)\phi^k. \tag{29}$$

Similarly, using the equation (27), (29) and $\widetilde{D}_\delta K^i = -\widetilde{\triangle}\phi^i - {K_{ab}}^i {K^{ab}}_j \phi^j + g(R(e_a, n_j), e^a, n^i)\phi^j$ (see [5]), we obtain for $\psi^i = K^i$,

$$\begin{aligned} \widetilde{D}_\delta \widetilde{\nabla}_b K^i &= \widetilde{\nabla}_b \widetilde{D}_\delta K^i - (D_\delta {\omega_b}^{ij} - \widetilde{\nabla}_b \gamma^{ij})K_j \\ &= -\widetilde{\nabla}_b \widetilde{\Delta}\phi^i - {K_{cd}}^i K^{cdj} \widetilde{\nabla}_b \phi_j - \widetilde{\nabla}_b({K_{cd}}^i K^{cdj})\phi_j + \widetilde{\nabla}_b g(R(e_a, n^j)e^a, n^i)\phi_j \\ &+ g(R(e_a, n^j)e^a, n^i)\widetilde{\nabla}_b \phi_j + {K_{bc}}^i \widetilde{\nabla}^c \phi^j K_j - {K_{bc}}^j \widetilde{\nabla}^c \phi^i K_j - g(R(n_k, e_b)n^j, n^i)\phi^k K_j. \end{aligned} \tag{30}$$

On the other hand, rewriting the symplectic potential as $\Psi^b = \sqrt{-\gamma}h^b$, with $h^b = \frac{1}{2}K^j K_j \phi^b + \phi_i \widetilde{\nabla}^b K^i - K_i \widetilde{\nabla}^b \phi^i$, it is easy to verify that,

$$\begin{aligned} D_\delta \Psi_b &= \widetilde{D}_\delta(\sqrt{-\gamma}h_b) \\ &= \left[\widetilde{D}_\delta \sqrt{-\gamma}\right] h_b \\ &+ \sqrt{-\gamma}\left[\widetilde{D}_\delta(\frac{1}{2}K^j K_j \phi_b) + \widetilde{D}_\delta \widetilde{\nabla}_b K^i \phi_i + \widetilde{\nabla}_b K_i \widetilde{D}_\delta \phi^i - \widetilde{D}_\delta K^i \widetilde{\nabla}_b \phi_i - K_i \widetilde{D}_\delta \widetilde{\nabla}_b \phi^i\right] \\ &= \sqrt{-\gamma}[(K_i K^i K^j \phi_j + K_i(-\widetilde{\triangle}\phi^i + g(R(e_a, n_j)e^a, n^i)\phi^j - K^{ab^i} K_{abj}\phi^j))\phi_b \\ &+ \frac{1}{2}K^j K_j {K_{bc}}^i \phi_i \phi^c - \frac{1}{2}K^j K_j \phi_i \widetilde{\nabla}_b \phi^i \\ &- K^j \phi_j K_i \widetilde{\nabla}_b \phi^i - \widetilde{\nabla}_b \widetilde{\Delta}\phi^i \phi_i + 2{K_{cd}}^i K^{cdj} \phi_i \widetilde{\nabla}_b \phi_j + K^j \phi_j \phi_i \widetilde{\nabla}_b K^i \\ &- 2g(R(e_a, n^j)e^a, n^i)\phi_i \widetilde{\nabla}_b \phi_j - 2{K_{bc}}^j \phi_j \widetilde{\nabla}^c \phi^i K_i + \widetilde{\Delta}\phi^i \widetilde{\nabla}_b \phi_i \\ &- 2K_i {K_{bc}}^i \widetilde{\nabla}^c \phi^j \phi_j - 2g(R(n_k, e_b)n^j, n^i)\phi^k K_j \phi_i], \end{aligned} \tag{31}$$

where we have used the equations (24), (28) and (30). In this manner, and in concordance with equation (31), we can see that

$$\begin{aligned} \delta\Psi^a &= D_\delta \Psi^a = \widetilde{D}_\delta(\gamma^{ab}\Psi_b) \\ &= \widetilde{D}_\delta \gamma^{ab} \Psi_b + \gamma^{ab} \widetilde{D}_\delta \Psi_b = \sqrt{-\gamma} j'^a, \end{aligned} \tag{32}$$

where

$$
\begin{aligned}
j'^a &= (K_i K^i K^j \phi_j + K_i(-\widetilde{\Delta}\phi^i + g(R(e_b, n_j)e^b, n^i)\phi^j - K^{cd^i} K_{cdj}\phi^j))\phi^a \\
&- \frac{1}{2}K^j K_j K^{aci}\phi_i\phi_c - \frac{1}{2}K^j K_j \phi_i \widetilde{\nabla}^a \phi^i \\
&+ \phi_i \widetilde{\nabla}^a \widetilde{\Delta}\phi^i + \widetilde{\Delta}\phi^i \widetilde{\nabla}^a \phi_i + 2K_{cd}{}^i K^{cdj}\phi_i \widetilde{\nabla}^a \phi_j \\
&- 2g(R(e_b, n^j)e^b, n^i)\phi_i \widetilde{\nabla}^a \phi_j - 2g(R(n_k, e^a)n^j, n^i)\phi^k K_j \phi_i - 2K^{abj}\phi_j \widetilde{\nabla}_b K^i \phi_i \\
&- 2K_i K^{abi}\widetilde{\nabla}_b \phi^j \phi_j + K^j \phi_j \phi_i \widetilde{\nabla}_b K^i - K^j \phi_j K_i \widetilde{\nabla}_b \phi^i,
\end{aligned} \tag{33}
$$

considering again that ϕ^a is a diffeomorphism on the worldvolume and therefore it can be gauged away ($\phi^a = 0$), we obtain finally

$$
\begin{aligned}
j'^a &= -\frac{1}{2}K^j K_j \phi_i \widetilde{\nabla}^a \phi^i + \phi_i \widetilde{\nabla}^a \widetilde{\Delta}\phi^i + \widetilde{\Delta}\phi^i \widetilde{\nabla}^a \phi_i + 2K_{cd}{}^i K^{cdj}\phi_i \widetilde{\nabla}^a \phi_j \\
&- 2g(R(e_b, n^j)e^b, n^i)\phi_i \widetilde{\nabla}^a \phi_j - 2g(R(n_k, e^a)n^j, n^i)\phi^k K_j \phi_i - 2K^{abj}\phi_j \widetilde{\nabla}_b K^i \phi_i \\
&- 2K_i K^{abi}\widetilde{\nabla}_b \phi^j \phi_j + K^j \phi_j \phi_i \widetilde{\nabla}_b K^i - K^j \phi_j K_i \widetilde{\nabla}_b \phi^i.
\end{aligned} \tag{34}
$$

If we take the special case of extremal surfaces ($K^i = 0$) in a curved background, we find

$$
j'^a = \phi_i \widetilde{\nabla}^a \widetilde{\Delta}\phi^i + \widetilde{\Delta}\phi^i \widetilde{\nabla}^a \phi_i + 2K_{cd}{}^i K^{cdj}\phi_i \widetilde{\nabla}^a \phi_j - 2g(R(e_b, n^j)e^b, n^i)\phi_i \widetilde{\nabla}^a \phi_j, \tag{35}
$$

which corresponds exactly to the current found in the Section III (equation (26)), using the self-adjoint operators method.

With these results, we can define a two-form in terms of j'^a, given in (35), that will be the symplectic structure that we require,

$$
\omega \equiv \int_\Sigma \sqrt{-\gamma} j'^a d\Sigma_a. \tag{36}
$$

where Σ is a Cauchy p-surface.

In this manner, we can see that our symplectic structure is an exact two-form because it comes from a exterior derivative of the global symplectic potential on Z (equation (32)) and it is in particular closed due to that δ is nilpotent, this is

$$
\delta\omega = \int_\Sigma \delta(\delta\Psi^a) d\Sigma_a = 0, \tag{37}
$$

therefore, we can see that it is advantageous to identify a symplectic potential from total divergences terms, that in the literature are neglected, but it is relevant on the phase space Z, since it generates our symplectic structure ω by means of a direct exterior derivative.

On the other hand, because of $\nabla_a j^a = 0$, we can use the Stokes theorem in (36) and see that ω is independent on the choice of Σ, this is $\omega_\Sigma = \omega_{\Sigma'}$. It will be a very important property of ω, since it allows us to establish

a connection between functions and Hamiltonian vector fields on Z; this subject will be considered in the future works.

Now, we shall prove that the symplectic structure that we have found (equation (36)) is invariant under infinitesimal spacetime diffeomorphism, which corresponds to the gauge directions of the theory on Z [1, 4, 6]. For this purpose, let us consider first the indeterminacy directions of the global symplectic potential on Z, and we notice from equation (32) that there are more than one symplectic potential, since δ is nilpotent, this is

$$\sqrt{-\gamma}j'^a = \delta(\Psi^a + \delta\eta^a), \tag{38}$$

where η^a is an arbitrary (worldvolume) field. On the other hand, we know that an infinitesimal spacetime diffeomorphism (δX^μ) induces a infinitesimal diffeomorphism on the worldvolume ($\delta\xi^a$), this is

$$\delta X^\mu = \epsilon^a \partial_a X^\mu = e^\mu_a \delta\xi^a, \tag{39}$$

where $e^\mu_a = \frac{\partial X^\mu}{\partial \xi^a}$ and ξ^a are the world-volume coordinates. In this manner, in (38) we can identify that, in particular, $\eta^a = \xi^a$, then the indeterminacy directions of the symplectic potential, contain, in particular, the gauge directions of the theory, therefore we have showed that ω is a no-degenerate two-form on Z. Furthermore, since Z is the set of solutions of equations of motion, it let $\widehat{Z}$ be the space of solutions modulo gauge transformations or quotient space $\widehat{Z} = Z/G$, where G is the group of spacetime diffeomorphisms; then we have that j'^a has vanishing components along the G orbits and therefore ω too.

VIII. Conclusions and prospects

As we have seen, the arguments of total divergences for the theories under study are identified as symplectic potentials, that are not relevant in the dynamics of the system, but are very important in the corresponding Witten covariant phase space, since they generate geometrical structures for [DNG] and [QEC] theories, confirming the results previously obtained in [4] for the former, and creating a symplectic structure for the later, that we will use in the future for identify the canonically conjugate variables, construct, for example, the corresponding Poisson brackets, find relevant symmetries, and study the quantization aspects for [QEC] theory. [QEC].

It is important to mention that the treatment in this paper for [QEC] p-branes is general, and contains the particular case of [QEC] string, in this manner, we have the necessary elements for study the quantization aspects of a different system (in this case [QEC] string) to that we commonly find in the literature, namely [DNG] string. For this aim, we need the symplectic structure that we have constructed, and solve the equa-

tions of motion, equation (10), which is crucial in the study of such aspects.

As we know, to solve the equations of motion, (10), for p-branes is very difficult; however, $K^i = 0$ is a subset of solutions of such equations, and in the literature we find the solutions to $K^i = 0$ corresponding to extremal surfaces, for [DNG] string, that are well know. In this manner, taking as particular case [QCE] string, we can use the same solutions to the equations of motion for [DNG] string, as a subset of solutions to equations (10) and the symplectic structure that we have constructed, equation (36), to study in an explicit way the quantization aspects for [QCE] strings, leaving it as a future work.

In addition, as a future work, we will show that finding a symplectic potential as in the presented work, we can identify the contributions of the topological terms in a canonical scheme, which is completely unknown in the literature [14].

Acknowledgements

This work was supported by CONACYT. The author wants to thank R. Cartas-Fuentevilla for drawing my interest to the study of branes.

Appendix

GEOMETRY OF THE EMBEDDING AND THEIR DEFORMATIONS

I. The embedding

The D-dimensional brane dynamics is usually given by a oriented timelike worldsheet m described by the embedding functions $x^\mu = X^\mu(\xi^a)$, $\mu = 0, ..., N-1$ and $a = 0, ..., D$, in a N-dimensional ambient spacetime M endowed with the metric $g_{\mu\nu}$. Such functions specify the coordinates of the brane, and the ξ^a correspond to internal coordinates on the worldsheet.

At each point of m, $e_a \equiv X^\mu_{,a}\partial_\mu \equiv e^\mu_a \partial_\mu$, generate a basis of tangent vectors to m; thus, the induced $(D+1)$-dimensional worldsheet metric is given by $\gamma_{ab} = e^\mu_a e^\nu_b g_{\mu\nu} = g(e_a, e_b)$. Furthermore, the $(N-D)$ vector fields n^i normal to m, are defined by

$$g(n^i, n^j) = \delta^{ij}, \quad g(e_a, n^i) = 0. \tag{40}$$

Tangential indices are raised and lowered by γ^{ab} and γ_{ab}, respectively, whereas normal vielbein indices by δ^{ij} and δ_{ij} respectively, and this fact will be used implicitly below. The collection of vectors $\{e_a, n^i\}$, which

can be used as a basis for the spacetime vectors, satisfies the generalized Gauss-Weingarten equation:

$$D_a e_b = \gamma_{ab}{}^c e_c - K_{ab}{}^i n_i, \quad D_a n_i = K_{ab}{}^i e^b + \omega_a{}^{ij} n_j,$$

where $D_a \equiv e^\mu_a D_\mu$ (D_μ is the torsionless covariant derivative associated with $g_{\mu\nu}$); thus, the connection coefficients $\gamma_{ab}{}^a$ compatible with γ_{ab} is given by $\gamma_{ab}{}^c = g(D_a e_b, e^c) = \gamma_{ba}{}^c$, and the ith extrinsic curvature of the worldsheet by $K_{ab}{}^i = -g(D_a e_b, n^i) = K_{ba}{}^i$. Similarly the extrinsic twist potential of the worldsheet is defined by $\omega_a{}^{ij} = g(D_a n^i, n^j) = -\omega_a{}^{ji}$. Such a potential allows us to introduce a worldsheet covariant derivative ($\widetilde{\nabla}_a$) defined on fields ($\Phi^i{}_j$) transforming as tensors under normal frame rotations:

$$\widetilde{\nabla}_a \Phi^i{}_j \equiv \nabla_a \Phi^i{}_j - \omega_a{}^{ik} \Phi_{kj} - \omega_{ajk} \Phi^{ik}, \tag{41}$$

where ∇_a is the (torsionless) covariant derivative associated with γ_{ab}.

II. Deformations of the intrinsic geometry

The deformation of the embedding is given by an arbitrary infinitesimal deformation δX^μ, decompose into its parts tangential and normal to the worldsheet

$$\delta X^\mu = e_a{}^\mu \phi^a + n_i{}^\mu \phi^i, \tag{42}$$

where

$$\phi^a = \delta X^\mu e^a{}_\mu \quad \text{and} \quad \phi^i = \delta X^\mu n^i{}_\mu, \tag{43}$$

however, in this scheme of deformations [5], the physically observable measure of the deformation of the embedding m, is given by the orthogonal projection of the infinitesimal spacetime variation $\xi^\mu \equiv \delta X^\mu = n^\mu_i \phi^i$, characterized by $N - D$ scalar fields ϕ^i, and the scalar fields $\phi^a = \delta X^\mu e^a{}_\mu$, are neglected because of it is identify as a diffeomorphism on the worldsheet.

Defining the vector field $\delta \equiv n_i \phi^i$, the displacement induced in the tangent basis $\{e_a\}$ along δ depends on ϕ^i and on their first derivatives:

$$D_\delta e_a = \beta_{ab} e^b + J_{ai} n^i, \tag{44}$$

where $D_\delta \equiv \delta^\mu D_\mu$, and

$$\beta_{ab} = g(D_\delta e_a, e_b) = K_{ab}{}^i \phi_i, \quad J_{ai} = g(D_\delta e_a, n_i) = \widetilde{\nabla}_a \phi_i; \tag{45}$$

similarly, the deformation in the induced metric on m is given by

$$D_\delta \gamma_{ab} = 2\beta_{ab} = 2K_{ab}{}^i \phi_i, \quad D_\delta \gamma^{ab} = -2\beta^{ab}. \tag{46}$$

For the case treated here, this is sufficient about the deformations of the intrinsic geometry.

III. Deformations of the extrinsic geometry

Introducing a covariant deformation derivative as $\widetilde{D}_\delta \Psi_i \equiv D_\delta \Psi_i - \gamma_i{}^j \Psi_j$, where $\gamma_{ij} = g(D_\delta n_i, n_j) = -\gamma_{ij}$, the covariant measure of the deformations of the quantities characterizing the extrinsic geometry are given by

$$D_s n_i = -J_{ai} e^a + \gamma_{ij} n^j, \quad \widetilde{D}_\delta n_i = -J_{ai} e^a = -(\widetilde{\nabla}_a \phi_i) e^a, \tag{47}$$

$$\widetilde{D}_\delta K_{ab}{}^i = -\widetilde{\nabla}_a \widetilde{\nabla}_b \phi^i + [K_{ac}{}^i K^c{}_{bj} - g(R(e_a, n_j)e_b, n^i)]\phi^j, \tag{48}$$

$$\widetilde{D}_\delta \omega_a{}^{ij} - \nabla_a \gamma^{ij} = D_\delta \omega_a{}^{ij} - \widetilde{\nabla}_a \gamma^{ij} = -K_{ab}{}^i \widetilde{\nabla}^b \phi^j + K_{ab}{}^j \widetilde{\nabla}^b \phi^i + g(R(n_k, e_a)n^j, n^i)\phi^k, \tag{49}$$

which depend on second derivatives of ϕ_i; the notation $g(R(Y_1, Y_2)Y_3, Y_4) = R_{\mu\nu\alpha\beta} Y_1^\nu Y_2^\mu Y_3^\alpha Y_4^\beta$ is used, where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor of spacetime. Other useful formulae and more details can be found directly in Ref. [5].

IV. The exterior calculus on Z

The space phase for [DNG] and [QEC] theories, is the space of solutions of equations (4) and (10) respectively and we call Z (see Section V). Any background quantity, such as those defined in Section II of this appendix, will be associated with zero-forms on Z [4].

On Z, the deformation operator δ acts as an exterior derivative, taking k-forms into $(k+1)$-forms, and it should satisfy

$$\delta^2 = 0, \tag{50}$$

and the Leibniz rule

$$\delta(AB) = \delta AB - A\delta B. \tag{51}$$

In this manner, δX^μ is the exterior derivative of the zero-form X^μ, and it be closed

$$\delta^2 X^\mu = 0. \tag{52}$$

Thus, since $\phi^i = \delta X^\mu n^i{}_\mu$ and $\phi^a = \delta X^\mu e^a{}_\mu$, corresponds to zero-forms on Z, and there are anticommutating objets: $\phi^i \phi^j = -\phi^j \phi^i$, $\phi^a \phi^b = -\phi^b \phi^a$. In general, the differential forms satisfy the Grassman algebra,

$AB = (-1)^{AB} BA$

It is remarkable to mention, that the covariant deformations operator D_δ (and subsequently $\widetilde{D}_\delta$) also works as an exterior derivative on Z, in the sense that maps k-forms into $(k+1)$-forms; however $\widetilde{D}_\delta^2$ does not vanish necessarily. In this manner, from equation (44) we can identify β_{ab} and j_{ai} as one- forms on Z.

Whit these preliminaries, it is very easy to prove the following [4]

$$D_\delta(\delta X^\mu) = 0, \tag{53}$$

and

$$\widetilde{D}_\delta(\phi^i) = 0. \tag{54}$$

Finally, using the equations (43), (44), (46), (51) and (53) we can calculate the exterior derivative of ϕ^a on Z, obtaining

$$\begin{aligned} \delta\phi^a = D_\delta(\gamma^{ab}\phi_b) &= (D_\delta \gamma^{ab})\phi_b + \gamma^{ab} D_\delta(\phi_b) \\ &= -2K^{abi}\phi_i\phi_b + \gamma^{ab}[D_\delta(\delta X^\mu)e_{b\mu} - \delta X^\mu D_\delta e_{b\mu}] \\ &= -2K^{abi}\phi_i\phi_b - \gamma^{ab}\delta X^\mu [K^{abi}\phi_i e_{b\mu} + \widetilde{\nabla}^a \phi^i n_{i\mu}] \\ &= -2K^{abi}\phi_i\phi_b - \gamma^{ab}[-K^{abi}\phi_i\phi_b - \widetilde{\nabla}^a\phi^i\phi_i] \\ &= -[K^{abi}\phi_i\phi_b + \phi_i\widetilde{\nabla}^a\phi^i]. \end{aligned} \tag{55}$$

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